S650 Assignment 1: Math Worksheet

This review covers basic mathematics that you will need for this class. Each section begins with Examples illustrating one way to solve the problem. You are to answer the Exercises. Answers are given for some of the problems that you are to solve; you need to provide the steps to get these answers. If you find this assignment to be trivial, just do it as quickly as possible. If you are having difficulties with a problem, find a text in basic algebra and review the appropriate topic. Then, come see me or a TA for further help.

Turn in your answers on a separate sheet of paper. Make sure you put your name on the upper right corner. Work neatly and be precise in your answers.

Part 1. Products and Quotients of Powers

Examples

1. $2^2 \times 2^4 = [2\times 2][2\times 2\times 2\times 2] = [2\times 2\times 2\times 2\times 2\times 2] = 2^{2+4} = 2^6$
2. $e^3 e^4 = [e\times e\times e][e\times e\times e\times e] = [e\times e\times e\times e\times e\times e\times e] = e^{3+4} = e^7$
3. $a^M a^N = [a\cdots(M)\cdots a][a\cdots(N)\cdots a] = [a\cdots(M+N)\cdots a] = a^{M+N}$
4. $\dfrac{e^5}{e^3} = \dfrac{e\times e\times e\times e\times e}{e\times e\times e} = e^{5-3} = e^2$
5. $\dfrac{a^M}{a^N} = \dfrac{[a\cdots(M)\cdots a]}{[a\cdots(N)\cdots a]} = a^{M-N}$

Exercises: Show the steps to solve each problem.

1. $2^5 \times 2^5 = 2^{10}$
2. $\dfrac{7^{19}}{7^{17}} = 7^2$
3. $e^9 \times e^{-3} = e^6$
4. $\dfrac{\beta^6\beta^7\beta^8}{\beta^2\beta^3\beta^4} = \beta^{12}$
5. $\dfrac{x^N}{x^k} = x^{N-k}$

Part 2. Natural Logarithms

Natural logarithms and exponentials are used extensively. A key reason is that they turn multiplication into addition, and addition is easier to work with than multiplication. Here are the basic principles. Every positive real number m can be written as $m=e^p$, where e=2.71828… If $m=e^p$, then we write $\ln m=p$. If we define $n=e^q$, then

$$mn = e^p e^q = e^{p+q}$$

$$\ln(m\times n) = \ln m + \ln n$$

$$\ln\left(\frac{m}{n}\right) = \ln m - \ln n$$

$$\ln\left(m^n\right) = n\ln m$$

<u>Examples</u>

1. $\ln 2 + \ln 3 = .693... + 1.0986... = 1.7917... = \ln(2 \times 3) = \ln(6)$
2. $\ln 2 - \ln 3 = .693... - 1.0986... = -.4054... = \ln(2/3) = \ln(.6666...)$
3. $\ln(e^b) = b \ln(e) = b \times 1 = b$
4. $\ln \frac{a}{b} + \ln \frac{b}{c} = [\ln a - \ln b] + [\ln b - \ln c] = \ln a - \ln c = \ln \frac{a}{c}$

<u>Exercises</u>: Show the steps to solve each problem.

1. $\ln 3 + \ln 4 = \ln 12$
2. $\ln 8 - \ln 4 = \ln 2$
3. $\ln(\alpha x^{\beta} \varepsilon) = \ln \alpha + \beta \ln x + \ln \varepsilon$
4. $\ln \frac{a}{c} - \ln \frac{b}{c} = \ln \frac{a}{b}$

Part 3. Vector Multiplication

<u>Example</u>

Let $\boldsymbol{\beta}' = (\beta_0 \quad \beta_1 \quad \beta_2)$ and $\mathbf{x} = (1 \quad x_1 \quad x_2)$, then $\mathbf{x}\boldsymbol{\beta} = \beta_0 1 + \beta_1 x_1 + \beta_2 x_2 = \beta_0 + \beta_1 x_1 + \beta_2 x_2$.

<u>Exercises</u>: Show the steps to solve each problem.

1. Let $\boldsymbol{\beta}' = (\alpha \quad \beta)$ and $\mathbf{x} = (1 \quad x)$, then $\mathbf{x}\boldsymbol{\beta} =$
2. Let $\boldsymbol{\beta}' = (\beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4)$ and $\mathbf{x} = (1 \quad x_1 \quad x_2 \quad x_3 \quad x_4)$, then $\mathbf{x}\boldsymbol{\beta} =$
3. Assume you have three independent variables; you decide what the substantive variables are. For example, one might be age. Show $\mathbf{x}$, $\boldsymbol{\beta}$, and $\mathbf{x}\boldsymbol{\beta}$.

Part 4. Expectation

The expectation can be thought of as the mean for the entire population.

1. For a discrete variable, the expectation can be computed as:
$$E(X) = \sum_x \Pr(X = x)x$$
2. If *X* and *Y* are random variables, and *a*, *b* and *c* are constants, then
$$E(a + bX + cY) = a + bE(X) + cE(Y)$$

<u>Examples</u>

1. If *X* has values 0 and 1 with probabilities 1/4 and 3/4, then
$$E(X) = \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{3}{4}\right) = \frac{3}{4}$$

Note that the mean is mixing the values of *X* where each value is weighted by its relative frequency.

2. Let $y_i = \alpha + \beta x_i + \varepsilon_i$. Then,

$$\begin{aligned} E(y_i) &= E(\alpha + \beta x_i + \varepsilon_i) \\ &= E(\alpha) + E(\beta x_i) + E(\varepsilon_i) \\ &= \alpha + \beta E(x_i) + E(\varepsilon_i) \end{aligned}$$

Exercises: Show the steps to solve each problem.
Define a head as 1 and a tail as 0

1. What is the expected value when you flip a fair coin?
2. If the probability of a head is .05 and a tail is .95, what is the expected value when you flip the coin?
3. If the probability of a head is *p* and a tail is *1-p*, what is the expected value when you flip the coin?

Part 5. Rescaling Variables

We will often want to use addition and multiplication to change the scale of a variable so that the mean becomes 0 and the variance 1. If $\mathrm{E}(x) = \mu$ and $Var(x) = \sigma^2$, then $\mathrm{E}\left(\frac{x-\mu}{\sigma}\right) = 0$ and $Var\left(\frac{x-\mu}{\sigma}\right) = 1$.

Example

Assume that $\mathrm{E}(y) = 4$ and $\mathrm{Var}(y) = 9$. Then, $\mathrm{Var}\left(\frac{y-4}{3}\right) = 1$ and $\mathrm{E}\left(\frac{y-4}{3}\right) = 0$.

Exercises: Show the steps to solve each problem.

1. Assume that $\mathrm{E}(y) = 2$ and $\mathrm{Var}(y) = 16$. Construct a variable with mean 0 and variance 1.
2. Assume that $\mathrm{E}(y) = \mu$ and $\mathrm{Var}(y) = \pi^2 / 3$. Construct a variable with mean 0 and variance 1.
3. Assume that $\mathrm{E}\ (x) =<$ day of your birth $>$ and $\mathrm{Var}(x) =<$ month of your birth $>$. Construct a variable y with mean 0 and variance 1.

Part 6. Sample space

A complete list of all possible outcomes of a random "experiment" is called sample space or probability space and is denoted by S

Example

A coin is tossed, and then tossed again. The sample space is defined as:
S={HH, HT, TH, TT}

Exercise

A six-sided die is tossed. The sample space is defined as:

A coin is flipped and a die is rolled at the same time. The sample space is defined as:

Part 7. Conditional Probability

Let E1 and E2 be any two events defined in a sample space S such that P(E1) > 0. The conditional probability of E2, assuming E1 has already occurred, is given by

$$P(E_2 \mid E_1) = \frac{P(E_2 \text{ and } E_1)}{P(E_1)}$$

Example
Let A denote the event "student is female" and let B denote the event "student is a freshman". In a class of 100 students suppose 60 are freshman, and suppose that 10 of the freshman are females. Find the probability that if I pick a freshman, it will be a girl.

We want to find P(A|B).
Since 10 out of 100 students are both freshman and female, then P(A and B) = 10/100
And, 60 out of the 100 students are freshman, so P(B) = 60/100
So

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{10/100}{60/100} = \frac{1}{6}$$

Exercise
What is the probability that the total of two dice will be greater than 8, given that the first die is a 6?
Hint: You will need to calculate the number of possible outcomes when you throw 2 dice.

Part 8. Independent Events
If the occurrence or non-occurrence of E1 does not affect the probability of occurrence of E2, then P(E2 | E1) = P(E2) and E1 and E2 are said to be independent events.

Let's consider "E1 and E2" as the event that "both E1 and E2 occur".
If E1 and E2 are independent events, then:
P(E1 and E2) = P(E1) × P(E2)

Example
A fair die is tossed twice. Find the probability of getting a 4 or 5 on the first toss and a 1, 2, or 3 in the second toss.

P(E1) = *P*(4 or 5) = 2/6 = 1/3
P(E2) = *P*(1, 2 or 3) = 3/6 = 1/2

They are independent events, so
P(E1 and E2) = P(E1) × P(E2) = 1/3 × 1/2 = 1/6

Exercise
Two balls are drawn at random from a bag containing 4 red and 3 black balls. The first ball is replaced before the second is drawn. What is the probability that both balls drawn are red?